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Transport Properties of the Continuous-Time Random Walk with a Long-Tailed Waiting-Time Density

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We derive asymptotic properties of the propagator $p(\mathbf{r}, t)$ of a continuous-time random walk (CTRW) in which the waiting time density has the asymptotic form $\psi(t) \sim T^{\alpha}/t^{\alpha+1}$ when $t \gg T$ and $0 < \alpha < 1$. Several cases are considered; the main ones are those that assume that the variance of the displacement in a single step of the walk is finite. Under this assumption we consider both random walks with and without a bias. The principal results of our analysis is that one needs two forms to characterize $p(\mathbf{r}, t)$, depending on whether r is large or small, and that the small-r expansion cannot be characterized by a scaling form, although it is possible to find such a form for large r. Several results can be demonstrated that contrast with the case in which $\langle t \rangle = \int_0^\infty \tau\psi(\tau)\, d\tau$ is finite. One is that the asymptotic behavior of $p(\mathbf{0}, t)$ is dominated by the waiting time at the origin rather than by the dimension. The second difference is that in the presence of a field $p(\mathbf{r}, t)$ no longer remains symmetric around a moving peak. Rather, it is shown that the peak of this probability always occurs at $\mathbf{r} = \mathbf{0}$, and the effect of the field is to break the symmetry that occurs when $\langle t \rangle < \infty$. Finally, we calculate similar properties, although in not such great detail, for the case in which the single-step jump probabilities themselves have an infinite mean.

KEY WORDS: Random walks; disordered media; transport properties.

1. INTRODUCTION

Approximations based on the continuous-time random walk[1] (CTRW) have been used in a number of investigations of transport properties in disordered systems, particularly those that can be posed in terms of hopping

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models. These arise in a natural way in the context of solid state physics[2–10] as well as in the analysis of chromatographic systems,[11,12] which are particular examples of a number of subjects of current interest in the general area of transport in disordered media. The CTRW is a random walk on a lattice in which the times between successive steps are assumed to be independent, identically distributed random variables. In order to describe transport properties of a CTRW on an infinite lattice, one requires a knowledge of the probability of being at site $\mathbf{r}$ at time t. This probability will be denoted by $p(\mathbf{r}, t)$. Generally, one is mainly interested in the long-time limit of $p(\mathbf{r}, t)$ because it is only in that limit that one can hope to find limiting laws independent of the detailed structure of the random walk. For example, it is known that when the times between successive steps have a finite average and the variance of each (random) displacement is finite the analytical form of $p(\mathbf{r}, t)$ tends toward a limiting Gaussian form as a function of $\mathbf{r}$ at sufficiently long times. In the present paper we will consider only the case of separable CTRWs; that is, CTRWs in which the displacement in a given step of the random walk and the time between successive steps are independent random variables. That is, if $p(\mathbf{r}, t)\,dt$ is the joint probability that a given step of the random walk is equal to $\mathbf{r}$ and the probability that the following step occurs after an interval of between t and $t+dt$ units of time, this function can be decomposed in the form

$$p(\mathbf{r}, t) = p(\mathbf{r})\,\psi(t) \tag{1}$$

In the present paper we will be interested in finding the asymptotic behavior of $p(\mathbf{r}, t)$ for large t, when $\psi(t)$ has a long-time tail in the sense that

$$\psi(t) \sim T^{\alpha}/t^{\alpha+1} \tag{2}$$

for $t \gg T$, where T is a constant with the dimensions of time. It is known that $p(\mathbf{r}, t)$ will not have a limiting Gaussian form when $\psi(t)$ has the property given in Eq. (2). Tunaley[13] and Shlesinger *et al.*[14] have both found formal expressions for the Fourier–Laplace transform (with respect to time) of $p(\mathbf{r}, t)$ but were not able to provide a useful, explicit, inversion of the formula for that function. Ball *et al.*[15] discussed the problem that is the subject of the present paper, evaluating the inversion integral by the method of steepest descents.[16] However, their analysis was partially in error, so that while they found the correct exponential behavior of $p(\mathbf{r}, t)$ for the unbiased random walk, the prefactor of the exponential is in error. Furthermore, the range of validity under which the approximation is expected to be a useful one is not defined.

We present the results of an analysis of the problem for four different cases. The two most important cases, ones for which we provide the most complete results, require that the second moments of the displacement in a single step remain finite. If we use the decomposition shown in Eq. (1), this is equivalent to the assumption that

$$\int_{-\infty}^{\infty} \mathbf{r}_i^2 p(\mathbf{r}, t)\, d^D\mathbf{r} < \infty, \qquad i = 1, 2,..., D \tag{3}$$

where D is the number of dimensions. We will show that when the bias, or, equivalently, the average displacement per step is equal to zero, one needs two separate approximations for $p(\mathbf{r}, t)$, one valid in the neighborhood of $\mathbf{r} = \mathbf{0}$, and the second valid for large values of $r^2 = \mathbf{r} \cdot \mathbf{r}$. The large-$r$ result, as will be seen, is found by the method of steepest descents. The final two cases to be considered are those in which the displacement probabilities as well as the $\psi(t)$ are asymptotically equivalent to a stable law.

2. APPROXIMATION FOR FINITE-VARIANCE RANDOM WALKS

2.1. Symmetric Random Walks

In the analysis to follow we denote the Laplace transform of a function of time, e.g., $f(t)$, by the same function with a caret and an argument s, so that, for example, $\mathscr{L}\{f(t)\} = \hat{f}(s)$. Equation (2) is equivalent, in the Laplace transform domain, to the result that $\hat{\psi}(s)$ can be approximated by $\hat{\psi}(s) \sim 1 - (sT)^\alpha$ for $\alpha < 1$ and $s \to 0$. Without loss of generality we can set $T = 1$, in which case the time t will be expressed in terms of dimensionless units τ defined by $\tau = t/T$. We will also make a slightly stronger assumption on the form of $\hat{\psi}(s)$ purely for the sake of mathematical convenience. Specifically, we will assume that $\hat{\psi}(s)$ is

$$\hat{\psi}(s) = 1/(s^\alpha + 1) \tag{4}$$

which is the transform of a legitimate probability density and clearly retains the desired limiting form as $s \to 0$. Properties of the lattice random walk will be reflected in terms of analytical properties of the structure function $\lambda(\boldsymbol{\theta})$, which is defined in terms of the single-step transition probabilities $p(\mathbf{j})$ as

$$\lambda(\boldsymbol{\theta}) = \sum_{\mathbf{j}} p(\mathbf{j}) \exp(i\mathbf{j} \cdot \boldsymbol{\theta}) \tag{5}$$

When the second moments of displacement are finite, the structure function can be expanded in a neighborhood of $\boldsymbol{\theta}=\mathbf{0}$ as

$$\lambda(\boldsymbol{\theta}) \sim 1 + i\boldsymbol{\mu}' \cdot \boldsymbol{\theta} - \frac{\boldsymbol{\theta}' \cdot \mathbf{V} \cdot \boldsymbol{\theta}}{2} \tag{6}$$

where $\boldsymbol{\mu}$ is a vector whose jth component is the average displacement in a single step along coordinate j, and $\mathbf{V}$ is the variance-covariance matrix. To further simplify our analysis, we consider only the specific case

$$\mathbf{V} = \nu \mathbf{I} \tag{7}$$

where ν is the second moment of displacement, common to all coordinates, and $\mathbf{I}$ is the unit matrix. When the random walk is symmetric, the transition probability for a single step satisfies $\boldsymbol{\mu}=\mathbf{0}$ and the variance is $\nu=\sigma^2$. It is also possible to analyze problems posed by variance-covariance matrices more general than that given in Eq. (7), but the interesting qualitative features of the analysis are all included in the simpler special case of Eq. (7).

The starting point for our analysis is the exact representation of $\hat{p}(\mathbf{r}, s)$ for a random walker initially at the origin:

$$\begin{aligned}\hat{p}(\mathbf{r}, s) &= \frac{1-\hat{\psi}(s)}{s(2\pi)^D} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\exp(-i\mathbf{r}\cdot\boldsymbol{\theta})}{1-\hat{\psi}(s)\,\lambda(\boldsymbol{\theta})}\, d^D\boldsymbol{\theta} \\ &= \frac{s^{\alpha-1}}{(2\pi)^D} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\exp(i\mathbf{r}\cdot\boldsymbol{\theta})}{s^\alpha + 1 - \lambda(\boldsymbol{\theta})}\, d^D\boldsymbol{\theta}\end{aligned} \tag{8}$$

in which we have made use of the definition given in Eq. (4). We will derive an expansion for $p(\mathbf{r}, \tau)$ in a neighborhood of the origin for large τ, but it is instructive to first consider the form taken by $p(\mathbf{0}, \tau)$ in the large-time limit. Three cases must be considered separately, $D=1$, $D=2$, and $D \geqslant 3$. In the first two of these, setting $s=0$ in the integrand leads to a formal divergence due to the singularity at the origin in θ space. Consider first $D=1$. Since, when s is set equal to 0 the divergence of the integral comes from the singularity of the integrand at $\theta=0$, we need only approximate the integrand carefully in the neighborhood of the origin by writing

$$s^\alpha + 1 - \lambda(\theta) \sim s^\alpha + \sigma^2\theta^2/2 \tag{9}$$

When the limits of integration on θ are extended to $\pm\infty$, the resulting integral converges for $s \neq 0$, but the contribution to the integral from values of θ^2 greater than π^2 is negligible in comparison with the contribution from the interval $(-\pi^2, \pi^2)$. Hence we may approximate to $\hat{p}(0, s)$ by

$$\hat{p}(0, s) \sim \frac{s^{\alpha-1}}{2\pi} \int_{-\infty}^{\infty} \frac{1}{s^\alpha + \sigma^2\theta^2/2}\, d\theta = \frac{s^{\alpha/2-1}}{\sigma\sqrt{2}} \tag{10}$$

which is equivalent to the asymptotic estimate

$$p(0, \tau) \sim [\sigma 2^{1/2} \Gamma(1-\alpha/2)\, \tau^{\alpha/2}]^{-1}, \qquad \tau \to \infty \tag{11}$$

A similar calculation for $D = 2$ leads to the result

$$p(\mathbf{0}, \tau) \sim \frac{\Gamma(1+\alpha)\sin(\pi\alpha)}{2\pi^2\sigma^2} \frac{\ln \tau}{\tau^\alpha} \tag{12}$$

When s is set equal to 0 in the last line of Eq. (8) the resulting integrals converge when $D \geqslant 3$, which implies the result

$$p(\mathbf{0}, \tau) \sim K_D \tau^{-\alpha} \tag{13}$$

where K_D is the constant

$$K_D = \frac{1}{(2\pi)^D \Gamma(1-\alpha)} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{d^D\boldsymbol{\theta}}{1-\lambda(\boldsymbol{\theta})} \tag{14}$$

The asymptotic form for $p(\mathbf{0}, \tau)$ is interesting because it indicates that for $D \geqslant 3$ the rate-limiting step in determining $p(\boldsymbol{\theta}, \tau)$ is the time taken by the random walker to first leave the origin. This should be contrasted to the behavior of CTRWs in which the mean time between successive steps of the random walk is finite, when $p(0, \tau)$ is asymptotically proportional to $\tau^{-\alpha D/2}$, except in $D = 2$ dimensions in which case a logarithm appears. When the average waiting time between steps is finite in dimensions $\geqslant 3$ the mean time spent by the random walker in all visits to the origin is finite. However, when the mean time between successive steps is infinite this can never be the case, since the average time spent at the origin during the initial sojourn at that point is necessarily infinite.

Let us next consider the nature of the remaining terms in the expansion of $p(\mathbf{r}, \tau)$ around the origin. These can be found by expanding the term $\exp(i\mathbf{r} \cdot \boldsymbol{\theta})$ in a Taylor series. The result of such an expansion follows closely the one for the one-dimensional case, hence we consider that case alone. A formal expansion of the exponential allows us to write

$$\hat{p}(x, s) \sim \frac{s^{\alpha-1}}{2\pi} \int_{-\pi}^{\pi} \frac{1 - x^2\theta^2/2 + x^4\theta^4/24 - \cdots}{s^\alpha + 1 - \lambda(\theta)}\, d\theta \tag{15}$$

We have already examined the behavior of the lowest order term, finding the result given in Eq. (11). A calculation of the coefficient of x^2 requires that we evaluate the integral

$$I_2(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\theta^2}{s^\alpha + 1 - \lambda(\theta)}\, d\theta \tag{16}$$

which converges when $s = 0$ since the θ^2 in the numerator just cancels the singularity in the denominator. Hence the asymptotic expansion of $p(x, \tau)$ in the neighborhood of $x = 0$ is

$$p(x, \tau) \sim p(0, \tau) - \frac{1}{\Gamma(1-\alpha)\, \tau^\alpha} \left(I_2 \frac{x^2}{2} - I_4 \frac{x^4}{24} + \cdots \right) \tag{17}$$

in which the integrals I_{2n} are the integrals

$$I_{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\theta^{2n}}{1 - \lambda(\theta)}\, d\theta \tag{18}$$

We note that Eq. (17) is a lowest order approximation in the sense that it is calculated from $I_2(0)$. Further contributions from the s dependence of $I_2(s)$ lead to lower order terms in τ which are not taken into account in the expansion shown in Eq. (17). The important feature of the expression given in Eq. (17) is the absence of scaling for small x, i.e., in the bracketed terms. The calculation in $D = 2$ dimensions leads to an expansion analogous to that given in Eq. (17) except that the I_{2n} of Eq. (18) are replaced by double integrals. One can carry out a similar calculation in three or more dimensions, finding that in the small-r, large-τ regime $p(\mathbf{r}, \tau)$ can be factorized, to lowest order, in the form

$$p(\mathbf{r}, \tau) \sim p(\mathbf{0}, \tau)\, f(\mathbf{r}) \tag{19}$$

where $f(\mathbf{r})$ is a power series whose coefficients are found in terms of D-dimensional analogues of the integrals in Eq. (18). Again we see that scaling relations cannot be valid in the small-r regime.

Let us next calculate the large-r approximation to $p(\mathbf{r}, \tau)$. We will express space coordinates in terms of the dimensionless distance $\rho = r/\sigma$. The starting point of the analysis is the expression for $\hat{p}(\rho, s)$ calculated following ref. 15:

$$\hat{p}(\rho, s) \sim \frac{(2^{1/2}\rho)^{1-D/2}\, s^{(\alpha/2)(1+D/2)}}{\pi^{D/2} \sigma^D s} K_{1-D/2}(2^{1/2} \rho s^{\alpha/2}) \tag{20}$$

where $K_{1-D/2}(x)$ is a Bessel function of the second kind of imaginary order.

We will be interested in the behavior of this function for large ρ, which allows us to use the asymptotic form of the Bessel function to find

$$\hat{p}(\rho, s) \sim \frac{[\rho(2\pi)^{1/2}]^{(1/2)(1-D)}}{\sigma^D 2^{1/2}} s^{(\alpha/4)(1+D)-1} \exp(-2^{1/2}\rho s^{\alpha/2}) \qquad (21)$$

This expression is to be substituted into the inversion integral for the Laplace transform

$$p(\rho, \tau) = \frac{1}{2\pi i} \int \hat{p}(\rho, s)\, e^{s\tau}\, ds \qquad (22)$$

where the integration is to be performed along a line parallel to the imaginary axis and to the right of all of the singularities of the integrand. Before proceeding to an asymptotic evaluation of this integral using the method of steepest descents, it is useful to change the variable of integration from s to v by the transformation

$$s = (2^{1/2}\rho/\tau)^{2/(2-\alpha)}\, v \qquad (23)$$

in which case the inversion integral for $p(\rho, \tau)$ is expressed as

$$p(\rho, \tau) = \frac{B}{2\pi i} \left(\frac{\rho^{1-D(1-\alpha)}}{\tau^{\alpha(1+D)/2}} \right)^{1/(2-\alpha)} \int v^{(\alpha/4)(1+D)-1} \exp[\Omega(v - v^{\alpha/2})]\, dv \qquad (24)$$

where B is a constant which cancels out of our later calculations, and Ω is the dimensionless combination of parameters

$$\Omega = (2\rho^2/\tau^\alpha)^{1/(2-\alpha)} \qquad (25)$$

We will apply the method of steepest descents to evaluate the integral in Eq. (24) when $\Omega \gg 1$ or, equivalently, when

$$\rho \gg \tau^{\alpha/2}/2 \qquad (26)$$

which, accordingly, defines the large-ρ regime. The principal contribution to the integral comes from the neighborhood of the extremum of the exponent in Eq. (24), which occurs at $v = v_0$, where

$$\gamma_0 = (\alpha/2)^{2/(2-\alpha)} \qquad (27)$$

Let $p_{\text{app}}(\rho, \tau)$ be the approximate value of the probability distribution obtained from the steepest descent calculation. We shall follow Daniels[16] in adopting as our final approximation not $p_{\text{app}}(\rho, \tau)$, but rather its normalized version in D dimensions:

$$q(\rho, \tau) = \rho^{D-1} p_{\text{app}}(\rho, \tau) \Big/ \int_0^\infty \rho^{D-1} p_{\text{app}}(\rho, \tau)\, d\rho \tag{28}$$

The normalization step causes the constant in Eq. (24), $B/(2\pi)$, to drop out of the expression for $q(\rho, \tau)$. If we define the constant w_0 by the relation

$$w_0 = (v_0^{\alpha/2} - v_0)\, 2^{1/(2-\alpha)} \tag{29}$$

then $q(\rho, \tau)$ takes the form

$$q(\rho, \tau) = \frac{w_0^{D/2} \rho^{D/(2-\alpha)-1}}{(2-\alpha)\, \Gamma(D/2)\, \tau^{D\alpha/[2(2-\alpha)]}} \exp\left\{ -w_0 \frac{\rho^{2/(2-\alpha)}}{\tau^{\alpha/(2-\alpha)}} \right\} \tag{30}$$

Higher order corrections can be calculated using the expansion given by Daniels.[16] We see that in the regime being analyzed the exponential term does have a scaling form in terms of the parameter ρ^2/τ^α.

2.2. Biased Random Walks

In this subsection we retain the assumption that the spatial variance of the random walk is finite, but now assume that the random walk is subjected to a bias which we quantify by requiring that the average displacement in a single step be equal to μ with $\mu = (\boldsymbol{\mu} \cdot \boldsymbol{\mu})^{1/2}$. In this situation there are two factors influencing the asymptotic form of the probability distribution of the end-to-end vector of the random walk. The first of these is expressed in terms of asymmetric transition probabilities that cause the random walker to prefer motion in a particular direction, and the second is the long-time tail assumed for $\psi(\tau)$, which tends to keep the random walker in place at a single site for long periods of time. In contrast to the more familiar case in which the average time between successive steps is finite, we will see that the asymptotic form of $p(\mathbf{r}, t)$ is not a symmetric function of bias-adjusted distance $\mathbf{r} - \mu t$.

As in the case of the unbiased random walk, we consider first the problem of finding the asymptotic behavior of $p(\mathbf{0}, \tau)$. We will show that in the presence of biased transition probabilities the function $p(\mathbf{0}, \tau)$ has the same asymptotic time dependence in any number of dimensions up to a multiplicative constant. Let us, for example, analyze the small-s behavior of the one-dimensional $\hat{p}(0, s)$ from Eq. (8). To do so, we exponentiate the denominator of the integrand by using the representation $u^{-1} =$

$\int_0^\infty \exp(-u\xi)\, d\xi$, followed by an interchange of the orders of integration. This allows us to express $\hat{p}(0, s)$ in the form of a Laplace transform as

$$\hat{p}(0, s) = \frac{s^{\alpha-1}}{2\pi} \int_0^\infty e^{-s^\alpha \xi}\, d\xi \int_{-\pi}^{\pi} e^{-\xi[1-\lambda(\boldsymbol{\theta})]}\, d\boldsymbol{\theta} \tag{31}$$

Only the limit $s \to 0$ interests us. In this regime, by using an Abelian theorem for Laplace transforms,[17] we can conclude that the only significant contribution to the integral over ξ can come from the limit $\xi \to \infty$. In the same limit the major contribution in the integral over θ comes from the neighborhood of $\theta = 0$, where now we may approximate $1 - \lambda(\theta)$ by $i\mu\theta$. The resulting integral can be evaluated in closed form, which yields

$$\hat{p}(0, s) \sim \frac{s^{\alpha-1}}{\pi\mu} \int_0^\infty e^{-s^\alpha \xi} \frac{\sin(\xi\pi\mu)}{\xi}\, d\xi \sim \frac{s^{\alpha-1}}{2\mu} \tag{32}$$

The equivalent asymptotic result in the time domain is

$$p(0, \tau) \sim [2\mu\Gamma(1-\alpha)]^{-1}\, \tau^{-\alpha} \tag{33}$$

at sufficiently large τ. A similar calculation suffices to establish that in any number of dimensions

$$\hat{p}(\mathbf{0}, s) \sim K_D s^{\alpha-1} \tag{34}$$

where K_D is a constant that depends on the dimension D and the bias μ. The result of this last equation implies that $p(0, t)$ at long times is

$$p(\mathbf{0}, \tau) \sim \frac{K_D}{\Gamma(1-\alpha)}\, \tau^{-\alpha} \tag{35}$$

That the asymptotic time dependence of $p(\mathbf{0}, \tau)$ should be independent of dimension is intuitively plausible. The bias transforms the model into an essentially one-dimensional one, independent of the dimension of the underlying space. When $D = 1$ one finds the intuitively plausible result that the probability that the random walker is found at the origin at time τ, $p(0, \tau)$, goes to zero more quickly than does the corresponding result for the isotropic random walk, since the latter probability falls off as $\tau^{-\alpha/2}$ at large times. This change in the power of τ in $p(0, \tau)$ is the only significant effect of the long-time tail form chosen for $\psi(t)$. Even when we raise the number of dimensions, the order of the exponent appearing in $p(0, \tau)$ does not increase as it does for the isotropic random walk, because the biasing field tends to deny the random walker access to sites not lying along the axis established by the field.

Turning now to the problem of finding an approximate form of $\hat{p}(\mathbf{r}, s)$ for $s \to 0$ and $\mathbf{r} \neq 0$, we can follow the steps leading to Eq. (20) to find that

$$\hat{p}(\mathbf{r}, s) = \frac{2s^{\alpha-1} \exp(\mathbf{r} \cdot \boldsymbol{\mu}/v)}{(2\pi v)^{D/2}} \left(\frac{r^2}{\mu^2 + 2vs^\alpha}\right)^{1/2 - D/4} K_{D/2-1}\left[\frac{\mathbf{r}}{v}(\mu^2 + 2vs^\alpha)^{1/2}\right] \tag{36}$$

Let us note that although Eq. (8) has the correct normalization property, approximations to the function $\hat{p}(\mathbf{r}, s)$ may no longer satisfy $\int_0^\infty \hat{p}(\mathbf{r}, s)\, d\mathbf{r} = 1/s$, in which case it is necessary to use a normalization condition such as that in Eq. (28). When $r/v \gg 1$ we can expand both term in brackets and the Bessel function, retaining only the lowest order terms in s. In this way we find the approximation

$$\hat{p}(\mathbf{r}, s) \sim \frac{\exp[(\boldsymbol{\mu} \cdot \mathbf{r} - \mu r)/v]}{\mu(2\pi v)^{(D-1)/2}} \left(\frac{r}{\mu}\right)^{(1-D)/2} s^{\alpha-1} \exp\left(-\frac{r}{\mu} s^\alpha\right) \tag{37}$$

It is interesting to observe that the vector $\boldsymbol{\mu}$ appears only in a single term in this formula, $\exp(\boldsymbol{\mu} \cdot \mathbf{r}/v)$, the remaining terms depending only the magnitude of the displacement vector $\mathbf{r}$. It is possible to evaluate the inversion integral by using the method of steepest descents. On comparing the approximation for biased random walks in Eq. (37) to that for unbiased random walks in Eq. (21), we see that the Laplace transform factor $s^{\alpha/2}$ that appears in the exponent in the latter equation is replaced by an exponent s^α when a nonzero bias exists. Following the analysis given for the unbiased random walk, we see that the steepest descents approximation will be useful when $r \gg \mu\tau^\alpha$. Provided that this condition holds, the approximation to $p(\mathbf{r}, \tau)$ obtained by the method of steepest descents is given by

$$\begin{aligned} p(\mathbf{r}, \tau) &\sim C(\tau)\, r^{(\alpha - D + \alpha D)/[2(1-\alpha)]} \\ &\quad \times \exp\left[\frac{(\boldsymbol{\mu} \cdot \mathbf{r} - \mu r)}{v} - \alpha^{1/(1-\alpha)}(\alpha^{-1} - 1)\left(\frac{r}{\mu\tau^\alpha}\right)^{1/(1-\alpha)}\right] \end{aligned} \tag{38}$$

where $C(\tau)$ is a normalization constant dependeing only on τ.

While it is difficult to examine the structure of $\hat{p}(\mathbf{r}, s)$ in generality, it is useful to consider particular cases for which the Laplace transform in Eq. (36) can be inverted in closed form. We therefore consider $D = 1$ with $\alpha = 1/2$. This choice of parameters leads to the expression

$$p(x, \tau) \sim C(\tau) \exp\left[\frac{\mu}{v}(x - |x|) - \frac{x^2}{4\mu^2\tau}\right] \tag{39}$$

where $C(\tau)$ is a normalization constant, which can be expressed in terms of the error function. It is evident from Eq. (39) that the maximum value of

$p(x, \tau)$ occurs at $x=0$ for all values of τ, and, furthermore, that the derivative with respect to x at $x=0$ is discontinuous. These indicate qualitative differences between the case of a finite waiting time between successive steps and the present case. The principal difference is that the time to make the first step away from the origin dominates the qualitative behavior of $p(x, \tau)$ when the first moment of $\psi(t)$ is infinite. One notable difference between the case of a finite $\langle t \rangle$ and the present one is that the shape of the peak is no longer symmetric around a traveling peak, but rather the symmetry of the peak is broken by an application of the field. Typical curves of $p(x, \tau)$ as a function of x are shown in Fig. 1. These resemble curves generated by Scher and Montroll[3] for a random walk on a finite lattice with periodic boundary conditions. When $\alpha = 1/2$ and $D = 3$ one readily shows from Eq. (36) that

$$p(\mathbf{r}, \tau) \sim \frac{C(\tau)}{r} \exp\left(-\frac{(\mu r - \boldsymbol{\mu} \cdot \mathbf{r})}{v} - \frac{r^2}{4\mu^2 \tau} \right) \tag{40}$$

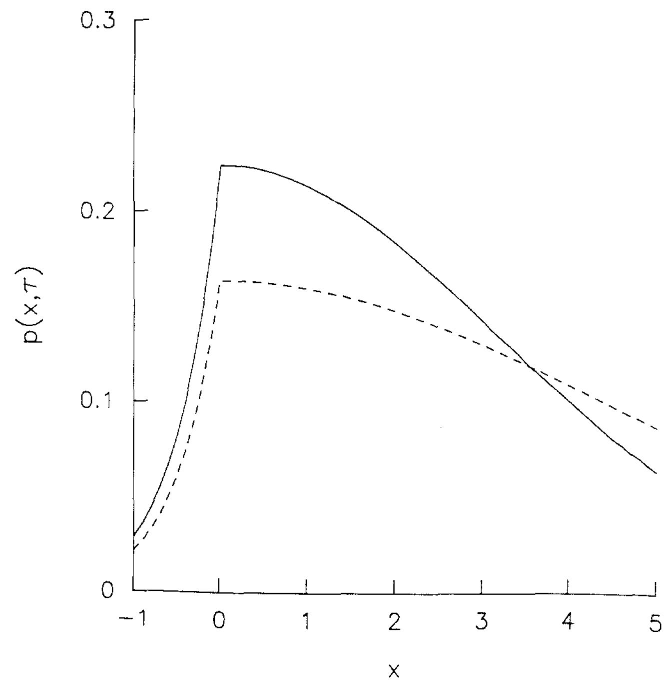

Fig. 1. Typical plots of $p(x, \tau)$ as a function of x for a 1D CTRW in the presence of a biasing field. The two curves correspond to different times, (—) $\tau = 10$ and (---) $\tau = 20$. Notice that the maximum of $p(x, \tau)$ remains at $x = 0$. In contrast, when $\int_0^\infty \tau\psi(\tau)\, d\tau$ is finite, the peak position moves as τ increases.

$C(\tau)$ is being found as a normalization factor. While there is now no discontinuity of $p(\mathbf{r}, \tau)$ with respect to $\mathbf{r}$, the qualitative behavior is quite similar to that found in $D=1$. When the angle between $\mathbf{r}$ and $\boldsymbol{\mu}$ is equal to 0 the term $\exp[-(\mu r - \boldsymbol{\mu}\cdot\mathbf{r})/v]$ is equal to 1; at angles other than 0 the exponential term is less than 1 since $\boldsymbol{\mu}\cdot\mathbf{r} = \mu r \cos\theta$, where θ is the angle between the vectors $\mathbf{r}$ and $\boldsymbol{\mu}$.

3. SYMMETRIC STABLE LAW RANDOM WALKS IN ONE DIMENSION

We consider next symmetric random walks in one dimension in which the displacement probabilities have the asymptotic property

$$p(j) \sim |j|^{-\beta-1}, \qquad 0<\beta\leqslant 1 \tag{41}$$

which implies that all integer moments of order greater than 0 diverge. The waiting time density $\psi(t)$ will be characterized by the property given in Eq. (2). It is known that for random walks satisfying the condition in Eq. (41) the structure function $\lambda(\theta)$ can be expanded in the neighborhood of $\theta=0$ as

$$\lambda(\theta) \sim 1 - |L\theta|^{\beta} \tag{42}$$

where L is a constant. The large-time limit implies that large distances are significant, allowing us to pass to a continuum limit. In this limit the constant L will have the dimensions of length, but we can set $L=1$ provided that we work in terms of dimensionless distances measured in units of L. Similarly, we will use the dimensionless distances measured in units of L. Similarly, we will use the dimensionless time τ introduced earlier, in which case we can set $T=1$. Rather than retaining the lattice structure, we will allow the line to be a continuum, denoting the dimensionless distance by the variable $y=x/L$.

Let us first calculate the asymptotic form of $p(0, \tau)$. We will show that the behavior of this function takes a different form depending on whether $\beta<1$ or $\beta>1$. To see this, let us return to the fundamental representation given in Eq. (8) for $D=1$. Consider first $\beta<1$, in which case setting $y=0$ leads to the approximation

$$\hat{p}(0, s) = \frac{s^{\alpha-1}}{\pi}\int_0^{\pi} \frac{d\theta}{s^{\alpha}+1-\lambda(\theta)} \tag{43}$$

We first calculate the exponent that characterizes the asymptotic falloff in time of the probability $p(0, \tau)$. Equation (43) indicates that different behavior is to be expected depending on whether $\beta < 1$, $= 1$, or > 1, since in the first case the integral converges when s is set equal to 0, while in the latter two cases it will diverge because of the singularity at $\theta = 0$. When $\beta < 1$ we find

$$\hat{p}(0, s) \sim \frac{s^{\alpha-1}}{\pi} \int_0^{\pi} \frac{d\theta}{1-\lambda(\theta)} \sim \frac{s^{\alpha-1}}{\pi} \int_0^{\pi} \frac{d\theta}{\theta^{\beta}} \tag{44}$$

Since the last integral on the right-hand side converges, the small-s behavior in Eq. (44) implies that large τ:

$$p(0, \tau) \sim K\tau^{-\alpha} \tag{45}$$

where the coefficient K is

$$K = \frac{1}{\pi\Gamma(1-\alpha)} \int_0^{\pi} \frac{d\theta}{1-\lambda(\theta)} \tag{46}$$

The asymptotic expression for $p(0, \tau)$ in Eq. (45) indicates that the controlling factor in the determination of the behavior of $p(0, \tau)$ for large τ is the initial sojourn at the origin. When $\beta > 1$ we cannot simply set $s = 0$ in Eq. (43) because the integral on the right-hand side of Eq. (44) diverges. In this case we consider the evaluation of Eq. (43) in the limit $s \to 0$ by setting $1 - \lambda(\theta) \sim \theta^{\beta}$ in the integrand, thereby finding, as an approximation,

$$\hat{p}(0, s) \sim \frac{s^{\alpha-1}}{\pi} \int_0^{\pi} \frac{d\theta}{s^{\alpha} + \theta^{\beta}} \tag{47}$$

On changing the variable of integration to v, where $\theta = s^{\alpha/\beta}$, we find that in the limit $s \to 0$,

$$\hat{p}(0, s) \sim \frac{s^{\alpha/\beta-1}}{\pi} \int_0^{\infty} \frac{dv}{v^{\beta}+1} \tag{48}$$

which indicates that

$$p(0, \tau) \sim \frac{\csc(\pi/\beta)}{\beta\Gamma(1-\alpha/\beta)} \tau^{-\alpha/\beta} \tag{49}$$

When $\beta = 1$ we find, starting from Eq. (47), that $\hat{p}(0, s) \sim \alpha(s^{\alpha-1}/\pi)\ln(1/s)$, which is equivalent to the asymptotic result

$$p(0, \tau) \sim \frac{\alpha}{\pi\Gamma(1-\alpha)} \frac{\ln \tau}{\tau^{\alpha}} \tag{50}$$

An expansion for $p(y, \tau)$ around $y=0$ can be found similar in form to that given in Eq. (17). The details of the derivation are quite similar to those used in finding that equation; hence, we only present the final result:

$$p(y, \tau) \sim p(0, \tau) - \frac{1}{\Gamma(1-\alpha)\,\tau^{\alpha}} \sum_{n=0}^{\infty} (-1)^{n+1} I_n \frac{y^{2n}}{(2n)!} \tag{51}$$

in which the I_n are just the constants given in Eq. (18). This result is valid independent of the value of β.

Let us next examine the asymptotic form of $\hat{p}(y, s)$ for general values of y. To do so, we will exponentiate the denominator of Eq. (8) in the one-dimensional case, finding, as a result, the integral representation

$$\hat{p}(y, s) = \frac{s^{\alpha-1}}{\pi} \int_0^{\infty} e^{-\xi s^{\alpha}}\, d\xi \int_0^{\pi} \cos(y\theta)\, e^{-\xi[1-\lambda(\theta)]}\, d\theta \tag{52}$$

i.e., $\hat{p}(y, s)$ is proportional to the Laplace transform of the function

$$g(\xi) = \int_0^{\pi} \cos(y\theta)\, e^{-\xi[1-\lambda(\theta)]}\, d\theta \tag{53}$$

Our interest in the value of $\hat{p}(y, s)$ for small s requires that we find the behavior of $g(\xi)$ for large ξ. In this limit the major contribution to $g(\xi)$ will come from the neighborhood of $\theta=0$. Hence we may replace Eq. (53) by

$$g(\xi) \sim \int_0^{\infty} \cos(y\theta)\, e^{-\xi\theta^{\beta}}\, d\theta \tag{54}$$

in which case, using an appropriate change of the variables of integration, we can rewrite Eq. (52) as the Laplace transform

$$\hat{p}(y, s) \sim s^{\alpha-1} \int_0^{\infty} \frac{e^{-\xi s^{\alpha}}}{\xi^{1/\beta}} Q_{\beta}(y\xi^{-1/\beta})\, d\xi \tag{55}$$

where $Q_{\beta}(x)$ is the tabulated integral[18]

$$Q_{\beta}(x) = \frac{1}{\pi} \int_0^{\infty} e^{-u^{\beta}} \cos(xu)\, du \tag{56}$$

Several asymptotic properties of $p(y, \tau)$ can be found from the integral representation given in the last line of Eq. (56), since the properties of $Q_{\beta}(x)$ are known. Let us consider the form of $\hat{p}(y, s)$ for large y. When $\beta<1$ the function $Q_{\beta}(x)$ can be represented by the convergent series

$$Q_{\beta}(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma(1+n\beta)}{n!\, x^{1+n\beta}} \sin\left(\frac{\pi n\beta}{2}\right) \tag{57}$$

On substituting this series into Eq. (55), interchanging the order of summation and integration, and performing the integration over ξ, we find

$$\hat{p}(y, s) \sim \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \Gamma(1+n\beta) \sin\left(\frac{\pi n\beta}{2}\right) \frac{1}{y^{1+n\beta} s^{1+n\alpha}} \tag{58}$$

When we invert the transform term by term we find

$$p(y, \tau) \sim \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma(1+n\beta)}{\Gamma(1+n\alpha)} \sin\left(\frac{\pi n\beta}{2}\right) \frac{\tau^{n\alpha}}{y^{1+n\beta}} \tag{59}$$

When $\beta > 1$ the series in Eq. (58) is asymptotically convergent and can therefore also be used for that regime. The series in Eq. (59) can only be regarded as being convergent in an asymptotic sense in the limit $y^{\beta}/\tau^{\alpha} \to \infty$. In contrast to the case of small y, the expansion given in this last equation does not depend on the value of β.

It is possible to calculate the asymptotic behavior of the expected number of distinct sites visited by a random walker in time t, $\langle S(t) \rangle$, for the cases treated here. The analysis follows that given in ref. 1, making use of the generating function for the expected number of distinct sites visited by a random walker in discrete time $\langle S_n \rangle$, $S(z) = z/[(1-z)^2 P(0; z)]$. It has been shown[1] that the Laplace transform of $\langle S(t) \rangle$ is just equal to $S(\hat{\psi}(s))$, which implies, using Tauberian methods, that for large t

$$\begin{aligned} S(t) &\sim t^{\alpha}, \qquad \beta < 1 \\ &\sim t^{\alpha/\beta}, \qquad 1 < \beta < 2 \end{aligned} \tag{60}$$

where we have omitted multiplicative constants. The transition in the exponent indicated in this last equation is clearly due to the fact that for small β the random walk essentially samples a new site on every step.

4. DISCUSSION

The CTRW has often been used as a model to approximate the parameters characterizing transport in a disordered medium.[2,3,9,19] The weak point in all such analyses has been the lack of a rigorous derivation of the form of the waiting time density $\psi(t)$ as well as a decision on whether one or more waiting time densities are needed to describe the transport properties of a given disordered system.[20] Some work on the derivation of an asymptotic form for this function for one-dimensional systems[9,21] suggests that if the CTRW description of transport is a valid one and a single $\psi(t)$ can be used to characterize transport in a disordered medium, it indeed has the property specified in Eq. (2). However, there is as yet no consensus in the literature on this subject as to when the CTRW description may be considered to be a useful one. It is possible, using the methods of this paper, to analyze properties of CTRWs with waiting-time densities having the asymptotic form

$$\psi(t) \sim \frac{1}{t[\ln(t/T)]^{\alpha+1}} \tag{61}$$

but these do not have a sufficient physical basis to warrant present study.

The CTRW has been suggested by a number of authors as a model for transport in a disordered medium.[2,3,9,11,12,19] Although the CTRW lacks the important feature of accounting for memory correctly, it has been shown[9,20] that by using a self-consistent equation for $\psi(t)$ one can derive a formula for asymptotic properties of transport dynamics. The principal result of the analysis yields the exponent of $p(r, t)$ in the form

$$\ln[p(r, t)] \sim -A\left(\frac{r}{t^{1/d_w}}\right)^{d_w/(d_w-1)} \tag{62}$$

where A is a constant. This expression agrees with a rigorous result for a one-dimensional system found by Stephens and Kariotis.[21] Equation (62) is in agreement with numerical data and scaling theories for diffusion on a variety of fractals exemplified by percolation clusters and the Sierpinsky gasket.[22]

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